

## Nonnull Curves with Constant Weighted Curvature in Lorentz-Minkowski Plane with Density

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**Abstract:** In this paper, the parametric expressions of spacelike and timelike curves with constant weighted curvature for some cases of  $a$  and  $b$  in Lorentz-Minkowski plane with density  $e^{ax+by}$  are obtained.

**Key words:** Lorentz-Minkowski plane, plane with density, weighted curvature

### 1. Introduction

In 2003, Gromov has introduced the notions of weighted mean curvature of an  $n$ -dimensional hypersurface and weighted curvature of a curve on manifolds with density as

$$H_\varphi = H - \frac{1}{n-1} \frac{d\varphi}{d\eta} \quad \text{and} \quad \kappa_\varphi = \kappa - \frac{d\varphi}{dN},$$

respectively [5]. Here,  $H$  is the mean curvature and  $\eta$  is the normal vector field of an  $n$ -dimensional hypersurface;  $\kappa$  is the curvature and  $N$  is the normal vector of the curve.

After these definitions, the differential geometry of the curves and hypersurfaces on manifolds with density in Euclidean, Minkowski and Galilean spaces has been started to be an important topic for geometers, physicists, economists and etc. For instance, in 2006 the authors have defined the weighted Gaussian curvature and they have given a generalization of Gauss-Bonnet formula for 2-dimensional differentiable manifold with density in [4]. In [11–13], F.Morgan has studied the manifolds with density, provided the generalizations of theorems of Myers and others to Riemannian manifolds with density and studied the Perelman's proof of the Poincaré conjecture, respectively.

The classification of constant weighted curvature curves in a plane with a log-linear density has been done in [7] and some other results, such as Fenchel's type theorem for the class of simple, closed, convex curves and the fact "the plane with density  $e^x$  contains no isoperimetric region" have been proved in [14]. In [10], Lopez has studied the minimal surfaces in Euclidean 3-space with a log-linear density  $\varphi(x, y, z) = \alpha x + \beta y + \gamma z$ , where  $\alpha$ ,  $\beta$  and  $\gamma$  are real numbers not all-zero. Also, Belarbi et al. have studied the surfaces in  $R^3$  with density and they have given some results in a Riemannian manifold  $M$  with density in [1] and [2], respectively.

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Furthermore, ruled and translation minimal surfaces in  $R^3$  with density  $e^z$ ; helicoidal surfaces in  $R^3$  with density  $e^{-x^2-y^2}$  and weighted minimal affine translation surfaces in Euclidean space with density have been studied in [6, 18, 19], respectively. Also, some types of surfaces have been studied by geometers in other spaces such as Minkowski 3-space and Galilean 3-space with density. For instance, a helicoidal surface of type  $I^+$  with prescribed weighted mean curvature and Gaussian curvature in Minkowski 3-space and weighted minimal translation surfaces in Minkowski 3-space with density  $e^z$  have been constructed in [15] and [16], respectively. In [17], weighted minimal translation surfaces in the Galilean 3-space with log-linear density has been classified and in [8], weighted minimal and weighted flat surfaces of revolution in Galilean 3-space with density  $e^{ax^2+by^2+cz^2}$  have been investigated.

Now, we'll recall some basic notions about curves in Lorentz-Minkowski plane.

Let  $\mathbb{L}^2$  be Lorentz-Minkowski plane defined as a space to be usual 2-dimensional vector space consisting of vectors  $\{(x^0, x^1) : x^0, x^1 \in \mathbb{R}\}$  but with a linear connection  $\nabla$  corresponding to its Minkowski metric  $g$  given by  $g(x, y) = -x^0y^0 + x^1y^1$ . Here, there are three categories of vector fields, namely,

*spacelike* if  $g(X, X) > 0$  or  $X = 0$ ,

*timelike* if  $g(X, X) < 0$ ,

*lightlike (null)* if  $g(X, X) = 0$ ,  $X \neq 0$ . In general, the type into which a given vector field  $X$  falls is called the *causal character* of  $X$ .

For a curve  $\alpha = (\alpha_1, \alpha_2) : I \subseteq \mathbb{R} \rightarrow \mathbb{R}^2$  in Lorentz-Minkowski plane, where  $I$  is some interval in  $\mathbb{R}$ , it is said that  $\alpha = \alpha(t)$  is spacelike (resp. timelike or lightlike) if the tangent vector  $\alpha'(t)$  is spacelike (resp. timelike or lightlike) for  $\forall t \in I$ . Now, let  $\alpha = (\alpha_1, \alpha_2)$  be a spacelike (or timelike) curve which is parametrized by arc-length; i.e.  $g(\alpha'(u), \alpha'(u)) = 1$  (or  $g(\alpha'(u), \alpha'(u)) = -1$ ), for  $\forall u \in I$ . For the curve  $\alpha = (\alpha_1, \alpha_2)$ , the tangent vector of it is  $T = \alpha' = (\alpha'_1, \alpha'_2)$  and we can choose the corresponding normal vector of  $\alpha$  as  $N = (\alpha'_2, \alpha'_1)$ . Here, it is obvious that the tangent vector  $T$  and normal vector  $N$  of  $\alpha$  have different causal characters. If we take  $g(T, T) = \epsilon$ , then we have  $g(N, N) = -\epsilon$  and the curve  $\alpha$  is spacelike for  $\epsilon = 1$  and timelike for  $\epsilon = -1$ . Also, the curvature of  $\alpha$  is the function of  $\kappa = \kappa(u)$  and it is given by

$$T'(u) = \kappa(u)N(u), \tag{1.1}$$

where

$$\kappa(u) = -\epsilon g(T'(u), N(u)) = \epsilon(x''(u)y'(u) - x'(u)y''(u)). \tag{1.2}$$

Furthermore, we have

$$N'(u) = \kappa(u)T(u). \tag{1.3}$$

For more details about Frenet dihedron of a curve in Lorentz-Minkowski plane, we refer to [3] and [9].

In the present study, we'll deal with spacelike and timelike curves in Lorentz-Minkowski plane with density and the aim of this study is to investigate the spacelike and timelike curves with constant weighted curvature in Lorentz-Minkowski plane with density  $e^{ax+by}$ .

## 2. Spacelike Curves with Constant Weighted Curvature in Lorentz-Minkowski Plane with Density

$e^{ax+by}$

In this section, we obtain the weighted curvature  $\kappa_\varphi$  of a spacelike curve  $\alpha$  in Lorentz-Minkowski plane with density  $e^{ax+by}$  and investigate the spacelike curves with constant weighted curvature for some cases of not all

zero constants  $a$  and  $b$ .

The weighted curvature  $\kappa_\varphi(u)$  of a spacelike curve  $\alpha(u) = (x(u), y(u))$  with arc-length parameter in Lorentz-Minkowski plane with density  $e^{ax+by}$  is obtained as

$$\kappa_\varphi(u) = \kappa(u) - \langle \nabla\varphi, N(u) \rangle = x''(u)y'(u) - x'(u)y''(u) + ay'(u) - bx'(u), \tag{2.1}$$

where  $N(u) = (y'(u), x'(u))$ . Since  $\alpha(u) = (x(u), y(u))$  is a spacelike curve with arc-length parameter, we can take

$$\begin{aligned} x'(u) &= \sinh(f(u)), \\ y'(u) &= \cosh(f(u)). \end{aligned} \tag{2.2}$$

So, from (2.1) and (2.2), the constant weighted curvature  $\kappa_\varphi(u)$  can be written as

$$\kappa_\varphi(u) = a \cosh(f(u)) - b \sinh(f(u)) + f'(u) = \lambda, \tag{2.3}$$

where  $\lambda \in \mathbb{R}$ .

Now, let us obtain the spacelike curves with constant weighted curvature  $\lambda$  according to some cases of constants  $a$  and  $b$ .

**2.1. The Case of "a = b"**

In this case, the equation (2.3) can be rewritten as

$$a \cosh(f(u)) - a \sinh(f(u)) + f'(u) = \lambda. \tag{2.4}$$

Using the definition of the hyperbolic functions in (2.4), we have

$$f'(u)e^{f(u)} = \lambda e^{f(u)} - a. \tag{2.5}$$

Now, we can find the spacelike curves by solving the equation (2.5) according to cases of constant  $\lambda$ .

**2.1.1. Solving equation (2.5) for  $\lambda = 0$ :**

Taking  $\lambda = 0$  in (2.5), we have

$$e^{f(u)} f'(u) = -a \tag{2.6}$$

and from (2.6), we get

$$d(e^{f(u)}) = -adu. \tag{2.7}$$

By integrating both sides of (2.7), we obtain that

$$e^{f(u)} = k_1 - au, \quad k_1 \in \mathbb{R}. \tag{2.8}$$

Hence, from (2.2) and (2.8), we have

$$\begin{aligned} x'(u) &= \frac{(k_1 - au)^2 - 1}{2(k_1 - au)}, \\ y'(u) &= \frac{(k_1 - au)^2 + 1}{2(k_1 - au)}. \end{aligned} \tag{2.9}$$

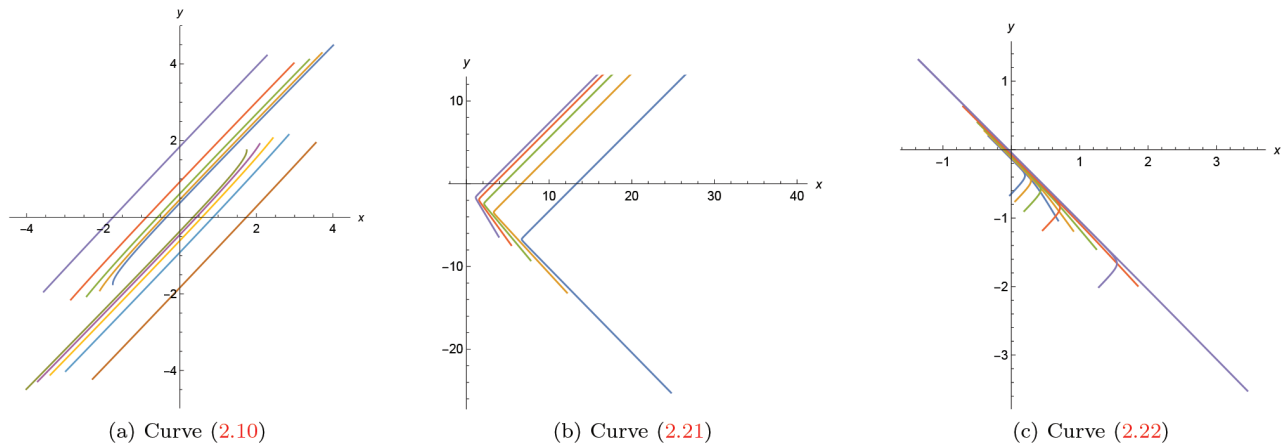
Thus, from (2.9), we can give the following Theorem:

**Theorem 2.1** *The spacelike curve  $\alpha(u)$  with vanishing weighted curvature in Lorentz-Minkowski plane with density  $e^{\alpha(x+y)}$  is*

$$\alpha(u) = \left( \frac{au(2k_1 - au) + 2 \ln(k_1 - au)}{4a} + k_2, \frac{au(2k_1 - au) - 2 \ln(k_1 - au)}{4a} + k_3 \right), \quad (2.10)$$

where  $k_1 - au > 0$  and  $k_i \in \mathbb{R}$ ,  $i = 1, 2, 3$ .

Figure 1 (a) shows this curve for  $a = -5, -4, -3, -2, -1, 1, 2, 3, 4, 5$ ,  $k_1 = 6$  and  $k_2 = k_3 = 0$ .



**Figure 1.** The spacelike curves with constant weighted curvature in Lorentz-Minkowski plane with density  $e^{\alpha(x+y)}$  for different values of constant  $a$ .

### 2.1.2. Solving equation (2.5) for $\lambda \neq 0$ :

Firstly, let's take  $f'(u) = 0$ . From equation (2.5), we have

$$e^{f(u)} = \frac{a}{\lambda}. \quad (2.11)$$

So, from (2.2) and (2.11), we get

$$\begin{aligned} x'(u) &= \frac{a^2 - \lambda^2}{2a\lambda}, \\ y'(u) &= \frac{a^2 + \lambda^2}{2a\lambda}. \end{aligned} \quad (2.12)$$

Now, let us take  $f'(u) \neq 0$ . Then from (2.5), we get

$$\frac{f'(u)e^{f(u)}}{e^{f(u)} - \frac{a}{\lambda}} = \lambda \quad (2.13)$$

and

$$\frac{d(e^{f(u)})}{e^{f(u)} - \frac{a}{\lambda}} = \lambda du. \quad (2.14)$$

By integrating both sides of (2.14), we have

$$\left| e^{f(u)} - \frac{a}{\lambda} \right| = e^{\lambda u + k_6}. \tag{2.15}$$

From (2.15), we can write

$$e^{f(u)} = \frac{a}{\lambda} + e^{\lambda u + k_6} \tag{2.16}$$

or

$$e^{f(u)} = \frac{a}{\lambda} - e^{\lambda u + k_6}. \tag{2.17}$$

Using (2.16) and (2.17) in (2.2), then we have

$$\begin{aligned} x'(u) &= \frac{\frac{a}{\lambda} + e^{\lambda u + k_6} - \frac{1}{\frac{a}{\lambda} + e^{\lambda u + k_6}}}{2}, \\ y'(u) &= \frac{\frac{a}{\lambda} + e^{\lambda u + k_6} + \frac{1}{\frac{a}{\lambda} + e^{\lambda u + k_6}}}{2} \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} x'(u) &= \frac{\frac{a}{\lambda} - e^{\lambda u + k_6} - \frac{1}{\frac{a}{\lambda} - e^{\lambda u + k_6}}}{2}, \\ y'(u) &= \frac{\frac{a}{\lambda} - e^{\lambda u + k_6} + \frac{1}{\frac{a}{\lambda} - e^{\lambda u + k_6}}}{2}, \end{aligned} \tag{2.19}$$

respectively. Hence, from (2.12), (2.18) and (2.19), we can state the following Theorem:

**Theorem 2.2** *The spacelike curves with non-zero constant weighted curvature  $\kappa_\varphi = \lambda$  in Lorentz-Minkowski plane with density  $e^{a(x+y)}$  are*

*i) the straight line of*

$$\alpha(u) = \left( \frac{a^2 - \lambda^2}{2a\lambda}u + k_4, \frac{a^2 + \lambda^2}{2a\lambda}u + k_5 \right), \tag{2.20}$$

whose slope is  $\frac{a^2 + \lambda^2}{a^2 - \lambda^2}$ . In the case of  $a = \lambda$ , the slope of lines is  $\infty$ , i.e. the lines are vertical, where  $\frac{a}{\lambda} > 0$  and  $k_i \in \mathbb{R}$ ,  $i = 4, 5$ ;

*ii)*

$$\alpha(u) = \left( \frac{au + e^{\lambda u + k_6}}{2\lambda} + \frac{\ln |ae^{-\lambda u} + \lambda e^{k_6}|}{2a} + k_7, \frac{au + e^{\lambda u + k_6}}{2\lambda} - \frac{\ln |ae^{-\lambda u} + \lambda e^{k_6}|}{2a} + k_8 \right), \tag{2.21}$$

where  $\frac{ae^{-\lambda u}}{\lambda} > -e^{k_6}$  and  $k_i \in \mathbb{R}$ ,  $i = 6, 7, 8$  or

*iii)*

$$\alpha(u) = \left( \frac{au - e^{\lambda u + k_6}}{2\lambda} + \frac{\ln |ae^{-\lambda u} - \lambda e^{k_6}|}{2a} + k_9, \frac{au - e^{\lambda u + k_6}}{2\lambda} - \frac{\ln |ae^{-\lambda u} - \lambda e^{k_6}|}{2a} + k_{10} \right), \tag{2.22}$$

where  $\frac{ae^{-\lambda u}}{\lambda} > e^{k_6}$  and  $k_i \in \mathbb{R}$ ,  $i = 6, 9, 10$ .

Figure 1 (b) shows the curve (2.21) for  $\lambda = 10$ ,  $a = 1, 2, 3, 4, 5$ ,  $k_6 = 11$  and  $k_7 = k_8 = 0$  and Figure 1 (c) shows the curve (2.22) for  $\lambda = 10$ ,  $a = 1, 2, 3, 4, 5$ ,  $k_6 = 1$  and  $k_9 = k_{10} = 0$ .

**2.2. The Case of "a ≠ 0, b = 0"**

In this case, the equation (2.3) can be rewritten as

$$a \cosh(f(u)) + f'(u) = \lambda. \tag{2.23}$$

From (2.23), we have

$$2f'(u)e^{f(u)} = a \left[ \frac{\lambda^2}{a^2} - 1 - \left( e^{f(u)} - \frac{\lambda}{a} \right)^2 \right]. \tag{2.24}$$

Now, we can obtain the spacelike curves by solving the equation (2.24) according to cases of constant λ.

**2.2.1. Solving equation (2.24) for λ = a:**

Firstly, let's take f'(u) = 0. From equation (2.24), we have

$$e^{f(u)} = 1. \tag{2.25}$$

So, from (2.2) and (2.25), we get

$$\begin{aligned} x(u) &= c_1, \\ y(u) &= u + c_2, \quad c_1, c_2 \in \mathbb{R}. \end{aligned} \tag{2.26}$$

Now, let us take f'(u) ≠ 0 in equation (2.24). Then, we have

$$\frac{2f'(u)e^{f(u)}}{(e^{f(u)} - 1)^2} = -a$$

and so,

$$\frac{2d(e^{f(u)})}{(e^{f(u)} - 1)^2} = -adu. \tag{2.27}$$

By integrating both sides of (2.27), we get

$$e^{f(u)} = \frac{2}{au - c_3} + 1. \tag{2.28}$$

So, from (2.2) and (2.28)

$$\begin{aligned} x'(u) &= \frac{(\frac{2}{au-c_3}+1)^2-1}{2(\frac{2}{au-c_3}+1)}, \\ y'(u) &= \frac{(\frac{2}{au-c_3}+1)^2+1}{2(\frac{2}{au-c_3}+1)}. \end{aligned} \tag{2.29}$$

Thus, from (2.2.1) and (2.29), we have

**Theorem 2.3** *The spacelike curves α(u) with constant weighted curvature κ<sub>φ</sub> = λ in Lorentz-Minkowski plane with density e<sup>ax</sup> for λ = a are*

*i) the straight line of*

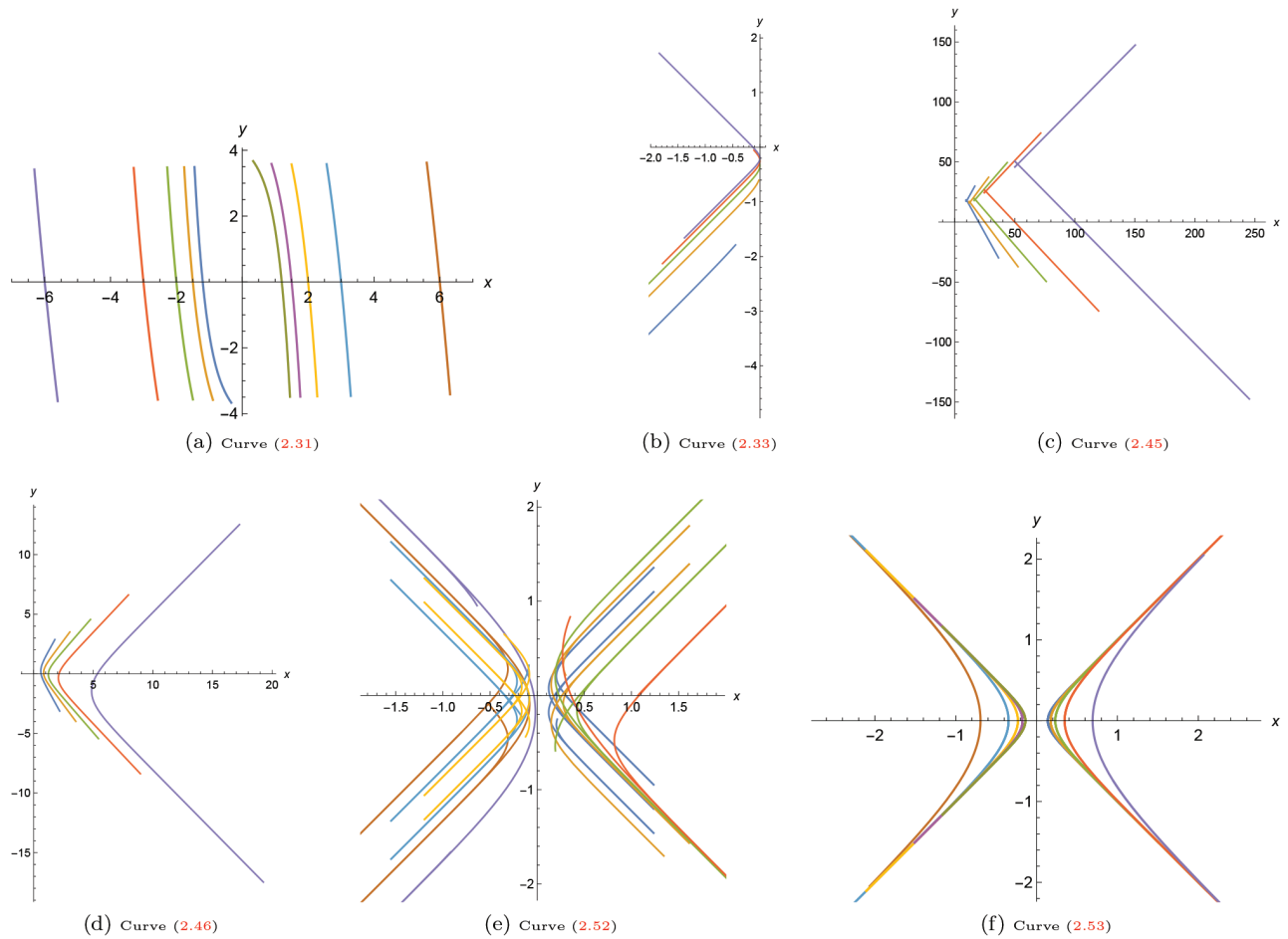
$$\alpha(u) = (c_1, u + c_2) \tag{2.30}$$

whose slope of lines is  $\infty$ , i.e. the lines are vertical and  $c_i \in \mathbb{R}$ ,  $i = 1, 2$  or **ii)**

$$\alpha(u) = \left( \frac{\ln((au - c_3)(au - c_3 + 2))}{a} + c_4, \frac{au + \ln\left(\frac{au - c_3}{au - c_3 + 2}\right)}{a} + c_5 \right), \quad (2.31)$$

where  $au - c_3 \in \mathbb{R} - [-2, 0]$  and  $c_i \in \mathbb{R}$ ,  $i = 3, 4, 5$ .

Figure 2 (a) shows the curve (2.31) for  $\lambda = a = -5, -4, -3, -2, -1, 1, 2, 3, 4, 5$ ,  $c_3 = 21$  and  $c_4 = c_5 = 0$ .



**Figure 2.** The spacelike curves with constant weighted curvature in Lorentz-Minkowski plane with density  $e^{ax}$  for different values of constant  $a$ .

**2.2.2. Solving equation (2.24) for  $\lambda = -a$ :**

With similar procedure in the case of  $\lambda = a$ , one can easily find that

$$\begin{aligned} x'(u) &= \frac{(\frac{2}{au-c_6}-1)^2-1}{2(\frac{2}{au-c_6}-1)}, \\ y'(u) &= \frac{(\frac{2}{au-c_6}-1)^2+1}{2(\frac{2}{au-c_6}-1)}. \end{aligned} \tag{2.32}$$

From (2.32), we can state the following Theorem:

**Theorem 2.4** *The spacelike curve  $\alpha(u)$  with constant weighted curvature  $\kappa_\varphi = \lambda$  in Lorentz-Minkowski plane with density  $e^{ax}$  for  $\lambda = -a$  is*

$$\alpha(u) = \left( \frac{\ln((au - c_6)(2 + c_6 - au))}{a} + c_7, -u + \frac{\ln\left(\frac{au-c_6}{2+c_6-au}\right)}{a} + c_8 \right), \tag{2.33}$$

where  $au - c_6 \in (0, 2)$  and  $c_i \in \mathbb{R}$ ,  $i = 6, 7, 8$ .

Figure 2 (b) shows this curve for  $\lambda = -a = -5, -4, -3, -2, -1$ ,  $c_6 = c_7 = c_8 = 0$ .

**2.2.3. Solving equation (2.24) for  $\frac{\lambda}{a} > 1$ :**

Firstly, let us assume that  $f'(u) = 0$ . From (2.24), we have

$$a \left[ \left( e^{f(u)} - \frac{\lambda}{a} \right)^2 + 1 - \frac{\lambda^2}{a^2} \right] = 0$$

and so,

$$e^{f(u)} = \frac{\lambda}{a} \pm \sqrt{\left(\frac{\lambda}{a}\right)^2 - 1}. \tag{2.34}$$

Thus, from (2.2) and (2.34), we have

$$\begin{aligned} x'(u) &= \frac{\frac{\lambda}{a} \pm \sqrt{\left(\frac{\lambda}{a}\right)^2 - 1} - \frac{1}{\frac{\lambda}{a} \pm \sqrt{\left(\frac{\lambda}{a}\right)^2 - 1}}}{2}, \\ y'(u) &= \frac{\frac{\lambda}{a} \pm \sqrt{\left(\frac{\lambda}{a}\right)^2 - 1} + \frac{1}{\frac{\lambda}{a} \pm \sqrt{\left(\frac{\lambda}{a}\right)^2 - 1}}}{2}. \end{aligned} \tag{2.35}$$

Now, let us take  $f'(u) \neq 0$ . From equation (2.24), we get

$$\begin{aligned} \frac{2f'(u)e^{f(u)}}{(e^{f(u)} - \frac{\lambda}{a})^2 + 1 - \frac{\lambda^2}{a^2}} &= -a \\ \Rightarrow \left( \frac{1}{e^{f(u)} - \frac{\lambda}{a} - \sqrt{\frac{\lambda^2}{a^2} - 1}} - \frac{1}{e^{f(u)} - \frac{\lambda}{a} + \sqrt{\frac{\lambda^2}{a^2} - 1}} \right) \frac{f'(u)e^{f(u)}}{\sqrt{\frac{\lambda^2}{a^2} - 1}} &= -a \\ \Rightarrow \left( \frac{1}{e^{f(u)} - \frac{\lambda}{a} - \sqrt{\frac{\lambda^2}{a^2} - 1}} - \frac{1}{e^{f(u)} - \frac{\lambda}{a} + \sqrt{\frac{\lambda^2}{a^2} - 1}} \right) d(e^{f(u)}) &= -a\sqrt{\frac{\lambda^2}{a^2} - 1}du. \end{aligned} \tag{2.36}$$



By integrating both sides of (2.36), we get

$$\left| \frac{e^{f(u) - \frac{\lambda}{a} - \sqrt{\frac{\lambda^2}{a^2} - 1}}}{e^{f(u) - \frac{\lambda}{a} + \sqrt{\frac{\lambda^2}{a^2} - 1}} \right| = e^{-a\sqrt{\frac{\lambda^2}{a^2} - 1}u + c_{11}}. \tag{2.37}$$

Thus, we can write

$$\frac{e^{f(u) - \frac{\lambda}{a} - \sqrt{\frac{\lambda^2}{a^2} - 1}}}{e^{f(u) - \frac{\lambda}{a} + \sqrt{\frac{\lambda^2}{a^2} - 1}} = e^{-a\sqrt{\frac{\lambda^2}{a^2} - 1}u + c_{11}} \tag{2.38}$$

or

$$\frac{e^{f(u) - \frac{\lambda}{a} - \sqrt{\frac{\lambda^2}{a^2} - 1}}}{e^{f(u) - \frac{\lambda}{a} + \sqrt{\frac{\lambda^2}{a^2} - 1}} = -e^{-a\sqrt{\frac{\lambda^2}{a^2} - 1}u + c_{11}}. \tag{2.39}$$

From (2.38) and (2.39), we have

$$e^{f(u)} = \frac{\left(\sqrt{\frac{\lambda^2}{a^2} - 1} - \frac{\lambda}{a}\right) e^{-a\sqrt{\frac{\lambda^2}{a^2} - 1}u + c_{11}} + \frac{\lambda}{a} + \sqrt{\frac{\lambda^2}{a^2} - 1}}{\left(1 - e^{-a\sqrt{\frac{\lambda^2}{a^2} - 1}u + c_{11}}\right)} \tag{2.40}$$

and

$$e^{f(u)} = \frac{\left(\frac{\lambda}{a} - \sqrt{\frac{\lambda^2}{a^2} - 1}\right) e^{-a\sqrt{\frac{\lambda^2}{a^2} - 1}u + c_{11}} + \frac{\lambda}{a} + \sqrt{\frac{\lambda^2}{a^2} - 1}}{\left(1 + e^{-a\sqrt{\frac{\lambda^2}{a^2} - 1}u + c_{11}}\right)}, \tag{2.41}$$

respectively. So, from (2.2) and (2.40), we obtain

$$\begin{aligned} x'(u) &= \frac{\left(\frac{(m-n)e^{amu+c_{11}+n+m}}{1-e^{amu+c_{11}}}\right)^2 - 1}{2\left(\frac{(m-n)e^{amu+c_{11}+n+m}}{1-e^{amu+c_{11}}}\right)}, \\ y'(u) &= \frac{\left(\frac{(m-n)e^{amu+c_{11}+n+m}}{1-e^{amu+c_{11}}}\right)^2 + 1}{2\left(\frac{(m-n)e^{amu+c_{11}+n+m}}{1-e^{amu+c_{11}}}\right)} \end{aligned} \tag{2.42}$$

and from (2.2) and (2.41),

$$\begin{aligned} x'(u) &= \frac{\left(\frac{(n-m)e^{-amu+c_{11}+n+m}}{1+e^{-amu+c_{11}}}\right)^2 - 1}{2\left(\frac{(n-m)e^{-amu+c_{11}+n+m}}{1+e^{-amu+c_{11}}}\right)}, \\ y'(u) &= \frac{\left(\frac{(n-m)e^{-amu+c_{11}+n+m}}{1+e^{-amu+c_{11}}}\right)^2 + 1}{2\left(\frac{(n-m)e^{-amu+c_{11}+n+m}}{1+e^{-amu+c_{11}}}\right)} \end{aligned} \tag{2.43}$$

where  $m = \sqrt{\frac{\lambda^2}{a^2} - 1}$  and  $n = \frac{\lambda}{a}$ . Hence, from (2.35), (2.42) and (2.43), we have

**Theorem 2.5** *The spacelike curves with constant weighted curvature  $\kappa_\varphi = \lambda$  in Lorentz-Minkowski plane with density  $e^{ax}$  for  $\frac{\lambda}{a} > 1$  are*

*i) the straight line of*

$$\alpha(u) = \left( \left( \frac{n \pm m - \frac{1}{n \pm m}}{2} \right) u + c_9, \left( \frac{n \pm m + \frac{1}{n \pm m}}{2} \right) u + c_{10} \right), \tag{2.44}$$

whose slope is  $\frac{(n \pm m)^2 + 1}{(n \pm m)^2 - 1}$  and  $c_i \in \mathbb{R}$ ,  $i = 9, 10$ ;

*ii)*

$$\alpha(u) = \left( \begin{array}{l} \frac{u}{2} \left( n + m - \frac{1}{n+m} \right) + \frac{\ln((1-e^{c_{11}-amu})(n(1-e^{c_{11}-amu})+m(1+e^{c_{11}-amu})))}{a} + c_{12}, \\ \frac{u}{2} \left( n + m + \frac{1}{n+m} \right) + \frac{\ln(\frac{1-e^{c_{11}-amu}}{n(1-e^{c_{11}-amu})+m(1+e^{c_{11}-amu}))})}{a} + c_{13} \end{array} \right), \tag{2.45}$$

where  $(1 - e^{c_{11}-amu})$  and  $(n(1 - e^{c_{11}-amu}) + m(1 + e^{c_{11}-amu}))$  have same signs and  $c_i \in \mathbb{R}$ ,  $i = 11, 12, 13$  or

*iii)*

$$\alpha(u) = \left( \begin{array}{l} \frac{u}{2} \left( n + m - \frac{1}{n+m} \right) + \left( \frac{\ln(1+e^{c_{11}-amu}) + \ln|n(1+e^{c_{11}-amu})+m(1-e^{c_{11}-amu})|}{a} \right) + c_{14}, \\ \frac{u}{2} \left( n + m + \frac{1}{n+m} \right) + \left( \frac{\ln(1+e^{c_{11}-amu}) - \ln|n(1+e^{c_{11}-amu})+m(1-e^{c_{11}-amu})|}{a} \right) + c_{15} \end{array} \right), \tag{2.46}$$

where  $c_i \in \mathbb{R}$ ,  $i = 11, 14, 15$ . Here, we take  $m = \sqrt{\frac{\lambda^2}{a^2} - 1}$ ,  $n = \frac{\lambda}{a}$ .

Figure 2 (c) shows the curve (2.45) for  $\lambda = 6$ ,  $a = 1, 2, 3, 4, 5$ ,  $c_{11} = 50$  and  $c_{12} = c_{13} = 0$  and Figure 2 (d) shows the curve (2.46) for  $\lambda = 6$ ,  $a = 1, 2, 3, 4, 5$ ,  $c_{11} = 1$  and  $c_{14} = c_{15} = 0$ .

**2.2.4. Solving equation (2.24) for  $\frac{\lambda}{a} < -1$ :**

From equation (2.24), we get

$$\frac{e^{f(u)} - \frac{\lambda}{a} - \sqrt{\frac{\lambda^2}{a^2} - 1}}{e^{f(u)} - \frac{\lambda}{a} + \sqrt{\frac{\lambda^2}{a^2} - 1}} = e^{-a\sqrt{\frac{\lambda^2}{a^2} - 1}u + c_{11}}. \tag{2.47}$$

With the similar calculations with the previous subsection, from (2.47), we have the equation (2.42). So,

**Theorem 2.6** *The spacelike curve with constant weighted curvature  $\kappa_\varphi = \lambda$  in Lorentz-Minkowski plane with density  $e^{ax}$  for  $\frac{\lambda}{a} < -1$  is given by (2.45).*

**2.2.5. Solving equation (2.24) for  $-1 < \frac{\lambda}{a} < 1$ :**

From equation (2.24), we have

$$\frac{2f'(u)e^{f(u)}}{(e^{f(u)} - \frac{\lambda}{a})^2 + 1 - \frac{\lambda^2}{a^2}} = -a$$

and so

$$\frac{2d(e^{f(u)})}{\left(\sqrt{1-\frac{\lambda^2}{a^2}}\right)^2 \left(\left(\frac{e^{f(u)}-\frac{\lambda}{a}}{\sqrt{1-\frac{\lambda^2}{a^2}}}\right)^2+1\right)} = -adu. \tag{2.48}$$

By integrating both sides of (2.48), we get

$$\frac{1}{\sqrt{1-\frac{\lambda^2}{a^2}}} 2 \arctan\left(\frac{e^{f(u)}-\frac{\lambda}{a}}{\sqrt{1-\frac{\lambda^2}{a^2}}}\right) = c_{16} - au \tag{2.49}$$

and from (2.49), we can write

$$e^{f(u)} = \sqrt{1-\frac{\lambda^2}{a^2}} \tan\left(\frac{\sqrt{1-\frac{\lambda^2}{a^2}}(c_{16}-au)}{2}\right) + \frac{\lambda}{a}. \tag{2.50}$$

So, from (2.2) and (2.50), we have

$$\begin{aligned} x'(u) &= \frac{\left(r \tan\left(\frac{r(c_{16}-au)}{2}\right)+n\right)^2-1}{2\left(r \tan\left(\frac{r(c_{16}-au)}{2}\right)+n\right)}, \\ y'(u) &= \frac{\left(r \tan\left(\frac{r(c_{16}-au)}{2}\right)+n\right)^2+1}{2\left(r \tan\left(\frac{r(c_{16}-au)}{2}\right)+n\right)}, \end{aligned} \tag{2.51}$$

where  $r = \sqrt{1-\frac{\lambda^2}{a^2}}$  and  $n = \frac{\lambda}{a}$ . Hence, from (2.51), we can state the following Theorem:

**Theorem 2.7** *The spacelike curve  $\alpha(u)$  with constant weighted curvature  $\kappa_\varphi = \lambda$  in Lorentz-Minkowski plane with density  $e^{ax}$  for  $-1 < \frac{\lambda}{a} < 1$  is*

$$\alpha(u) = \left( \begin{array}{l} \frac{nc_{16}+2\ln|\cos(\frac{r(c_{16}-au)}{2})|+2\ln|(n \cos(\frac{r(c_{16}-au)}{2})+r \sin(\frac{r(c_{16}-au)}{2}))|}{2a} + c_{17}, \\ \frac{n(2au-c_{16})+2\ln|\cos(\frac{r(c_{16}-au)}{2})|-2\ln|(n \cos(\frac{r(c_{16}-au)}{2})+r \sin(\frac{r(c_{16}-au)}{2}))|}{2a} + c_{18} \end{array} \right), \tag{2.52}$$

where  $\tan(\frac{r(c_{16}-au)}{2}) > -\frac{n}{r}$ ,  $r = \sqrt{1-\frac{\lambda^2}{a^2}}$ ,  $n = \frac{\lambda}{a}$  and  $c_i \in \mathbb{R}$ ,  $i = 16, 17, 18$ .

Figure 2 (e) shows this curve for  $\lambda = 1$ ,  $a = -5, -4, -3, -2, 2, 3, 4, 5$ ,  $c_{16} = 1$  and  $c_{17} = c_{18} = 0$ .

So, taking  $\lambda = 0$  in (2.52), we have:

**Corollary 2.8** *The spacelike curve  $\alpha(u)$  with vanishing weighted curvature  $\kappa_\varphi$  in Lorentz-Minkowski plane with density  $e^{ax}$  is given by*

$$\alpha(u) = \left( \begin{array}{l} \frac{\ln|\frac{\sin(c_{16}-au)}{2}|}{a} + c_{17}, \\ \frac{\ln|\cot(\frac{c_{16}-au}{2})|}{a} + c_{18} \end{array} \right), \tag{2.53}$$

where  $\tan(\frac{r(c_{16}-au)}{2}) > -\frac{n}{r}$  and  $c_i \in \mathbb{R}$ ,  $i = 16, 17, 18$ .

Figure 2 (f) shows this curve for  $a = -5, -4, -3, -2, -1, 1, 2, 3, 4, 5$ ,  $c_{16} = 1$  and  $c_{17} = c_{18} = 0$ .

**2.3. The case of "a = 0, b ≠ 0"**

In this case, the equation (2.3) can be rewritten as

$$f'(u) - b \sinh(f(u)) = \lambda. \tag{2.54}$$

So, from (2.54), we have

$$2f'(u)e^{f(u)} = b \left[ \left( e^f + \frac{\lambda}{b} \right)^2 - \left( \frac{\lambda^2}{b^2} + 1 \right) \right]. \tag{2.55}$$

Now, we can find the spacelike curves by solving the equation (2.55) according to cases of constant λ.

**2.3.1. Solving equation (2.55) for λ = 0:**

Firstly, let us assume that f'(u) = 0. Then, from equation (2.55) we have

$$e^{f(u)} = 1. \tag{2.56}$$

So, from (2.2) and (2.56), we get

$$\begin{aligned} x(u) &= d_1, \\ y(u) &= u + d_2, \quad d_1, d_2 \in \mathbb{R}. \end{aligned} \tag{2.57}$$

Now, let us take f'(u) ≠ 0. From equation (2.55),

$$2f'(u)e^{f(u)} = b \left[ \left( e^{f(u)} \right)^2 - 1 \right] \tag{2.58}$$

and so,

$$\left( \frac{1}{e^{f(u)} - 1} - \frac{1}{e^{f(u)} + 1} \right) d(e^{f(u)}) = bdu. \tag{2.59}$$

By integrating both sides of (2.59), we get

$$\left| \frac{e^{f(u)} - 1}{e^{f(u)} + 1} \right| = e^{bu+d_3}. \tag{2.60}$$

Thus, we can write

$$e^{f(u)} = \frac{1 + e^{bu+d_3}}{1 - e^{bu+d_3}} \tag{2.61}$$

or

$$e^{f(u)} = \frac{1 - e^{bu+d_3}}{1 + e^{bu+d_3}}. \tag{2.62}$$

Hence, using (2.61) and (2.62) in (2.2), we reach that

$$\begin{aligned} x'(u) &= \frac{\frac{1+e^{bu+d_3}}{1-e^{bu+d_3}} - \frac{1-e^{bu+d_3}}{1+e^{bu+d_3}}}{2}, \\ y'(u) &= \frac{\frac{1+e^{bu+d_3}}{1-e^{bu+d_3}} + \frac{1-e^{bu+d_3}}{1+e^{bu+d_3}}}{2} \end{aligned} \tag{2.63}$$

and

$$\begin{aligned} x'(u) &= \frac{\frac{1-e^{bu+d_3}}{1+e^{bu+d_3}} - \frac{1+e^{bu+d_3}}{1-e^{bu+d_3}}}{2}, \\ y'(u) &= \frac{\frac{1-e^{bu+d_3}}{1+e^{bu+d_3}} + \frac{1+e^{bu+d_3}}{1-e^{bu+d_3}}}{2}, \end{aligned} \tag{2.64}$$

respectively. Thus, from (2.3.1), (2.63) and (2.64) we can give the following Theorem:

**Theorem 2.9** *The spacelike curves with vanishing weighted curvature  $\kappa_\varphi$  in Lorentz-Minkowski plane with density  $e^{by}$  are*

*i) the straight line of*

$$\alpha(u) = (d_1, u + d_2), \tag{2.65}$$

whose slope of lines is  $\infty$ , i.e. the lines are vertical and  $d_i \in \mathbb{R}$ ,  $i = 1, 2$ ;

*ii)*

$$\alpha(u) = \left( \frac{\ln\left(\frac{e^{-bu}+e^{d_3}}{e^{-bu}-e^{d_3}}\right)}{b} + d_4, -\left(\frac{\ln(e^{-2bu} - e^{2d_3})}{b} + u\right) + d_5 \right), \tag{2.66}$$

where  $bu + d_3 < 0$  and  $d_i \in \mathbb{R}$ ,  $i = 3, 4, 5$  or

*iii)*

$$\alpha(u) = \left( \frac{\ln\left(\frac{e^{-bu}-e^{d_3}}{e^{-bu}+e^{d_3}}\right)}{b} + d_6, -\left(\frac{\ln(e^{-2bu} - e^{2d_3})}{b} + u\right) + d_7 \right), \tag{2.67}$$

where  $bu + d_3 < 0$  and  $d_i \in \mathbb{R}$ ,  $i = 3, 6, 7$ .

Figure 3 (a) and (b) show the curves (2.66) and (2.67), respectively, for  $b = -1, -2, -3, -4, -5$ ,  $d_3 = -1$  and  $d_4 = d_5 = d_6 = d_7 = 0$ .

**2.3.2. Solving equation (2.55) for  $\lambda \neq 0$ :**

Firstly, let's assume that  $f'(u) = 0$ . From equation (2.55), we get

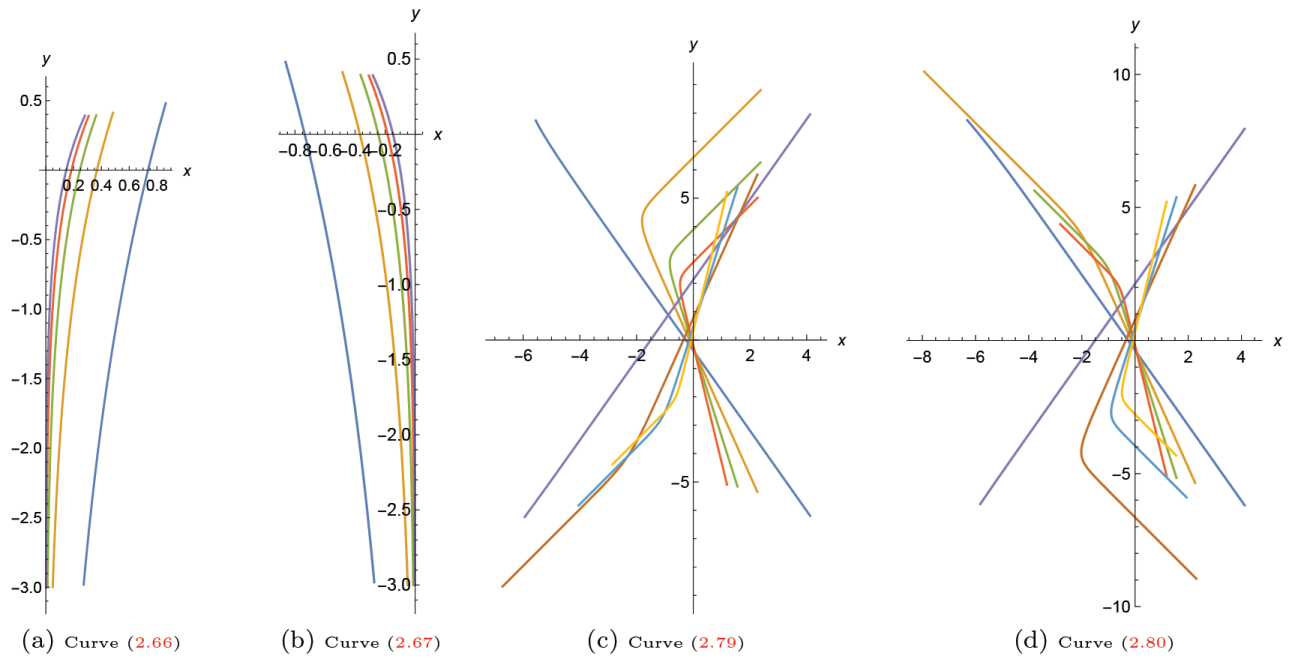
$$b \left[ \left( e^{f(u)} + \frac{\lambda}{b} \right)^2 - \left( \frac{\lambda^2}{b^2} + 1 \right) \right] = 0 \tag{2.68}$$

and so,

$$e^{f(u)} = \sqrt{\left(\frac{\lambda}{b}\right)^2 + 1} - \frac{\lambda}{b}. \tag{2.69}$$

Thus, from (2.2) and (2.69), we have

$$\begin{aligned} x'(u) &= \frac{\sqrt{\left(\frac{\lambda}{b}\right)^2 + 1} - \frac{\lambda}{b} - \frac{1}{\sqrt{\left(\frac{\lambda}{b}\right)^2 + 1} - \frac{\lambda}{b}}}{2}, \\ y'(u) &= \frac{\sqrt{\left(\frac{\lambda}{b}\right)^2 + 1} - \frac{\lambda}{b} + \frac{1}{\sqrt{\left(\frac{\lambda}{b}\right)^2 + 1} - \frac{\lambda}{b}}}{2}. \end{aligned} \tag{2.70}$$



**Figure 3.** The spacelike curves with constant weighted curvature in Lorentz-Minkowski plane with density  $e^{by}$  for different values of constant  $b$ .

Now, if  $f'(u) \neq 0$ , then the equation (2.55) can be rewritten as

$$\frac{2f'(u)e^{f(u)}}{\left(e^{f(u)} + \frac{\lambda}{b} - \sqrt{\frac{\lambda^2}{b^2} + 1}\right)\left(e^{f(u)} + \frac{\lambda}{b} + \sqrt{\frac{\lambda^2}{b^2} + 1}\right)} = b \tag{2.71}$$

and so,

$$\left(\frac{1}{e^{f(u)} + \frac{\lambda}{b} - \sqrt{\frac{\lambda^2}{b^2} + 1}} - \frac{1}{e^{f(u)} + \frac{\lambda}{b} + \sqrt{\frac{\lambda^2}{b^2} + 1}}\right) \frac{d(e^{f(u)})}{\sqrt{\frac{\lambda^2}{b^2} + 1}} = bdu. \tag{2.72}$$

By integrating both sides of (2.72), we get

$$\left|\frac{e^{f(u)} + \frac{\lambda}{b} - \sqrt{\frac{\lambda^2}{b^2} + 1}}{e^{f(u)} + \frac{\lambda}{b} + \sqrt{\frac{\lambda^2}{b^2} + 1}}\right| = e^{b\sqrt{\frac{\lambda^2}{b^2} + 1}u + d_{10}}. \tag{2.73}$$

Thus

$$\begin{aligned} \frac{e^{f(u)} + \frac{\lambda}{b} - \sqrt{\frac{\lambda^2}{b^2} + 1}}{e^{f(u)} + \frac{\lambda}{b} + \sqrt{\frac{\lambda^2}{b^2} + 1}} &= e^{b\sqrt{\frac{\lambda^2}{b^2} + 1}u + d_{10}} \\ \Rightarrow e^{f(u)} &= \frac{\left(\frac{\lambda}{b} + \sqrt{\frac{\lambda^2}{b^2} + 1}\right)e^{b\sqrt{\frac{\lambda^2}{b^2} + 1}u + d_{10}} + \sqrt{\frac{\lambda^2}{b^2} + 1} - \frac{\lambda}{b}}{1 - e^{b\sqrt{\frac{\lambda^2}{b^2} + 1}u + d_{10}}} \end{aligned} \tag{2.74}$$

or

$$\begin{aligned} \frac{ef(u) + \frac{\lambda}{b} - \sqrt{\frac{\lambda^2}{b^2} + 1}}{ef(u) + \frac{\lambda}{b} + \sqrt{\frac{\lambda^2}{b^2} + 1}} &= -e^{b\sqrt{\frac{\lambda^2}{b^2} + 1}u + d_{10}} \\ \Rightarrow e^{f(u)} &= \frac{\sqrt{\frac{\lambda^2}{b^2} + 1} - \frac{\lambda}{b} - (\frac{\lambda}{b} + \sqrt{\frac{\lambda^2}{b^2} + 1})e^{b\sqrt{\frac{\lambda^2}{b^2} + 1}u + d_{10}}}{1 + e^{b\sqrt{\frac{\lambda^2}{b^2} + 1}u + d_{10}}}. \end{aligned} \tag{2.75}$$

So, using (2.74) and (2.75) in (2.2), we have

$$\begin{aligned} x'(u) &= \frac{\left(\frac{(t+s)e^{bsu+d_{10}+s-t}}{1-e^{bsu+d_{10}}}\right)^2 - 1}{2\left(\frac{(t+s)e^{bsu+d_{10}+s-t}}{1-e^{bsu+d_{10}}}\right)}, \\ y'(u) &= \frac{\left(\frac{(t+s)e^{bsu+d_{10}+s-t}}{1-e^{bsu+d_{10}}}\right)^2 + 1}{2\left(\frac{(t+s)e^{bsu+d_{10}+s-t}}{1-e^{bsu+d_{10}}}\right)} \end{aligned} \tag{2.76}$$

and

$$\begin{aligned} x'(u) &= \frac{\left(\frac{s-t-(t+s)e^{bsu+d_{10}}}{1+e^{bsu+d_{10}}}\right)^2 - 1}{2\left(\frac{s-t-(t+s)e^{bsu+d_{10}}}{1+e^{bsu+d_{10}}}\right)}, \\ y'(u) &= \frac{\left(\frac{s-t-(t+s)e^{bsu+d_{10}}}{1+e^{bsu+d_{10}}}\right)^2 + 1}{2\left(\frac{s-t-(t+s)e^{bsu+d_{10}}}{1+e^{bsu+d_{10}}}\right)}, \end{aligned} \tag{2.77}$$

respectively, where  $s = \sqrt{\frac{\lambda^2}{b^2} + 1}$  and  $t = \frac{\lambda}{b}$ .

So, from (2.70), (2.76) and (2.77), we have

**Theorem 2.10** *The spacelike curves with non-zero constant weighted curvature  $\kappa_\varphi = \lambda$  in Lorentz-Minkowski plane with density  $e^{by}$  are*

i) *The straight line of*

$$\alpha(u) = \left( \left( \frac{s-t-\frac{1}{s-t}}{2} \right) u + d_8, \left( \frac{s-t+\frac{1}{s-t}}{2} \right) u + d_9 \right), \tag{2.78}$$

whose slope of lines is  $\frac{(s-t)^2+1}{(s-t)^2-1}$  and  $d_i \in \mathbb{R}$ ,  $i = 8, 9$ ;

ii)

$$\alpha(u) = \left( \begin{aligned} &\frac{u}{2} \left( \frac{1}{t+s} - t - s \right) - \frac{1}{b} \left( \ln(e^{-bsu} - e^{d_{10}}) - \ln(e^{d_{10}}(t+s) + (s-t)e^{-bsu}) \right) + d_{11}, \\ &\frac{-u}{2} \left( \frac{1}{t+s} + t + s \right) - \frac{1}{b} \left( \ln(e^{-bsu} - e^{d_{10}}) + \ln(e^{d_{10}}(t+s) + (s-t)e^{-bsu}) \right) + d_{12} \end{aligned} \right), \tag{2.79}$$

where  $bsu + d_{10} < 0$  and  $d_i \in \mathbb{R}$ ,  $i = 10, 11, 12$  or

iii)

$$\alpha(u) = \left( \begin{array}{l} \frac{u}{2} \left( \frac{1}{t+s} - t - s \right) - \frac{1}{b} \left( \ln |e^{-bsu} - e^{d_{10}}| - \ln |(s-t)e^{-bsu} - e^{d_{10}}(t+s)| \right) + d_{13}, \\ \frac{-u}{2} \left( \frac{1}{t+s} + t + s \right) - \frac{1}{b} \left( \ln |e^{-bsu} + e^{d_{10}}| + \ln |(s-t)e^{-bsu} - e^{d_{10}}(t+s)| \right) + d_{14} \end{array} \right), \quad (2.80)$$

where  $e^{bsu+d_{10}} < \frac{s-t}{s+t}$  and  $d_i \in \mathbb{R}$ ,  $i = 10, 13, 14$ . Here we take  $s = \sqrt{\frac{\lambda^2}{b^2} + 1}$  and  $t = \frac{\lambda}{b}$ .

Figure 3 (c) and (d) show the curves (2.79) and (2.80), respectively, for  $\lambda = 1$ ,  $b = -4, -3, -2, -1, 1, 2, 3, 4$ ,  $d_{10} = -10$  and  $d_{11} = d_{12} = d_{13} = d_{14} = 0$ .

### 3. Timelike curves with constant weighted curvature in Lorentz-Minkowski plane with density

$e^{ax+by}$

In this section, we obtain the weighted curvature  $\kappa_\varphi$  of a timelike curve  $\beta$  in Lorentz-Minkowski plane with density  $e^{ax+by}$  and investigate the timelike curves with constant weighted curvature for some cases of not all zero constants  $a$  and  $b$  with the same procedure in the previous section.

The weighted curvature  $\kappa_\varphi(u)$  of a timelike curve  $\beta(u) = (x(u), y(u))$  with arc-length parameter in Lorentz-Minkowski plane with density  $e^{ax+by}$  is obtained as

$$\kappa_\varphi = x'(u)y''(u) - x''(u)y'(u) + ay'(u) - bx'(u). \quad (3.1)$$

Since  $\beta(u) = (x(u), y(u))$  is a timelike curve with arc-length parameter, we can take

$$\begin{aligned} x'(u) &= \cosh(f(u)), \\ y'(u) &= \sinh(f(u)). \end{aligned} \quad (3.2)$$

So, from (3.1) and (3.2), the constant weighted curvature  $\kappa_\varphi$  can be written as

$$\kappa_\varphi = f'(u) + a \sinh(f(u)) - b \cosh(f(u)) = \lambda, \quad (3.3)$$

where  $\lambda \in \mathbb{R}$ .

Now, let us obtain the timelike curves with constant weighted curvature  $\lambda$  according to some cases of constants  $a$  and  $b$ .

#### 3.1. The case of "a = b"

In this case, the equation (3.3) can be rewritten as

$$f'(u) + a \sinh(f(u)) - a \cosh(f(u)) = \lambda. \quad (3.4)$$

Using the definition of the hyperbolic functions in (3.4), we have

$$f'(u)e^{f(u)} = a + \lambda e^{f(u)}. \quad (3.5)$$

Now, we can find the timelike curves by solving the equation (3.5) according to cases of constant  $\lambda$ .

Thus, we have the following Theorems which can be obtained with the same procedure in Subsection 2.1.

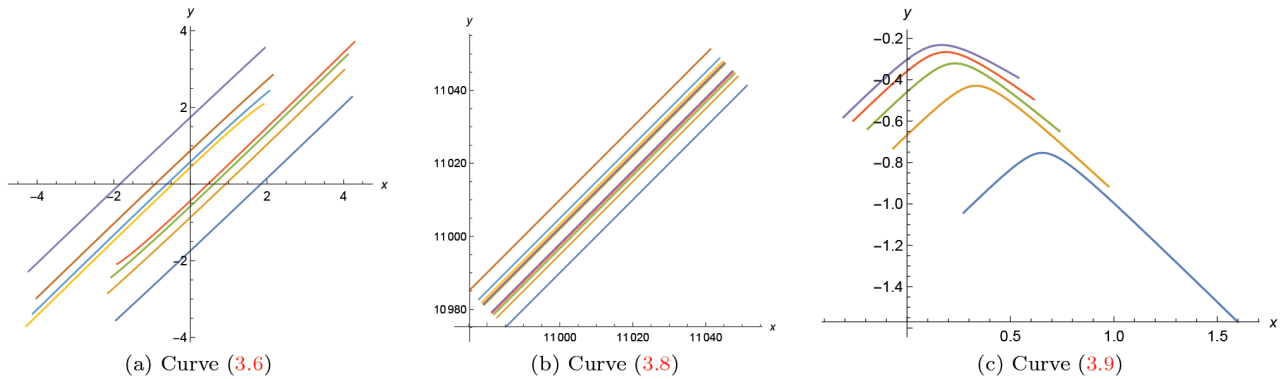


**Theorem 3.1** *The timelike curve  $\beta(u)$  with vanishing weighted curvature in Lorentz-Minkowski plane with density  $e^{a(x+y)}$  is*

$$\beta(u) = \left( \frac{au(au + 2l_1) + 2 \ln(au + l_1)}{4a} + l_2, \frac{au(au + 2l_1) - 2 \ln(au + l_1)}{4a} + l_3 \right), \tag{3.6}$$

where  $au + l_1 > 0$  and  $l_i \in \mathbb{R}$ ,  $i = 1, 2, 3$ .

Figure 4 (a) shows this curve for  $a = -5, -4, -3, -2, -1, 1, 2, 3, 4, 5$ ,  $l_1 = 6$  and  $l_2 = l_3 = 0$ .



**Figure 4.** The timelike curves with constant weighted curvature in Lorentz-Minkowski plane with density  $e^{a(x+y)}$  for different values of constant  $a$ .

**Theorem 3.2** *The timelike curves with non-zero constant weighted curvature  $\kappa_\varphi = \lambda$  in Lorentz-Minkowski plane with density  $e^{a(x+y)}$  are*

i) *the straight line of*

$$\beta(u) = \left( \frac{-a^2 - \lambda^2}{2a\lambda}u + l_4, \frac{\lambda^2 - a^2}{2a\lambda}u + l_5 \right), \tag{3.7}$$

whose slope is  $\frac{a^2 - \lambda^2}{a^2 + \lambda^2}$ . In the case of  $a = -\lambda$ , the slope of lines is 0, i.e. the lines are horizontal, where  $\frac{a}{\lambda} < 0$  and  $l_i \in \mathbb{R}$ ,  $i = 4, 5$ ;

ii)

$$\beta(u) = \left( \frac{e^{\lambda u + l_6} - au}{2\lambda} + \frac{\ln|\lambda e^{l_6} - ae^{-\lambda u}|}{2a} + l_7, \frac{e^{\lambda u + l_6} - au}{2\lambda} - \frac{\ln|\lambda e^{l_6} - ae^{-\lambda u}|}{2a} + l_8 \right), \tag{3.8}$$

where  $\frac{ae^{-\lambda u}}{\lambda} < e^{l_6}$  and  $l_i \in \mathbb{R}$ ,  $i = 6, 7, 8$  or

iii)

$$\beta(u) = \left( \frac{\ln|ae^{-\lambda u} + \lambda e^{l_6}|}{2a} - \frac{e^{\lambda u + l_6} + au}{2\lambda} + l_9, -\frac{\ln|ae^{-\lambda u} + \lambda e^{l_6}|}{2a} - \frac{e^{\lambda u + l_6} + au}{2\lambda} + l_{10} \right), \tag{3.9}$$

where  $-e^{l_6} > \frac{ae^{-\lambda u}}{\lambda}$  and  $l_i \in \mathbb{R}$ ,  $i = 6, 9, 10$ .

Figure 4 (b) shows the curve (3.8) for  $\lambda = 1, a = -5, -4, -3, -2, -1, 1, 2, 3, 4, 5, l_6 = 10$  and  $l_7 = l_8 = 0$  and Figure 4 (c) shows the curve (3.9) for  $\lambda = 10, a = 1, 2, 3, 4, 5, l_6 = -1$  and  $l_9 = l_{10} = 0$ .

**3.2. The case of "a = 0, b ≠ 0"**

In this case, the equation (3.3) can be rewritten as

$$f'(u) - b \cosh(f(u)) = \lambda. \tag{3.10}$$

From (3.10), we have

$$2f'(u)e^{f(u)} = b \left[ \left( e^{f(u)} + \frac{\lambda}{b} \right)^2 + 1 - \frac{\lambda^2}{b^2} \right]. \tag{3.11}$$

Now, we can obtain the timelike curves by solving the equation (3.11) according to cases of constant  $\lambda$ .

So, we have the following Theorems which can be obtained with the same procedure in Subsection 2.2.

**Theorem 3.3** *The timelike curves  $\beta(u)$  with constant weighted curvature  $\kappa_\varphi = \lambda$  in Lorentz-Minkowski plane with density  $e^{by}$  for  $\lambda = -b$  are*

*i) the straight line of*

$$\beta(u) = (u + h_1, h_2) \tag{3.12}$$

*whose slope of lines is 0, i.e. the lines are horizontal and  $h_i \in \mathbb{R}, i = 1, 2$  or*

*ii)*

$$\beta(u) = \left( \frac{bu + \ln \left( \frac{bu+h_3-2}{bu+h_3} \right)}{b} + h_4, -\frac{\ln((bu+h_3)(bu+h_3-2))}{b} + h_5 \right), \tag{3.13}$$

*where  $bu + h_3 \in \mathbb{R} - [0, 2]$  and  $h_i \in \mathbb{R}, i = 3, 4, 5$ .*

Figure 5 (a) shows the curve (3.13) for  $\lambda = -b = -5, -4, -3, -2, -1, 1, 2, 3, 4, 5, h_3 = 20$  and  $h_4 = h_5 = 0$ .

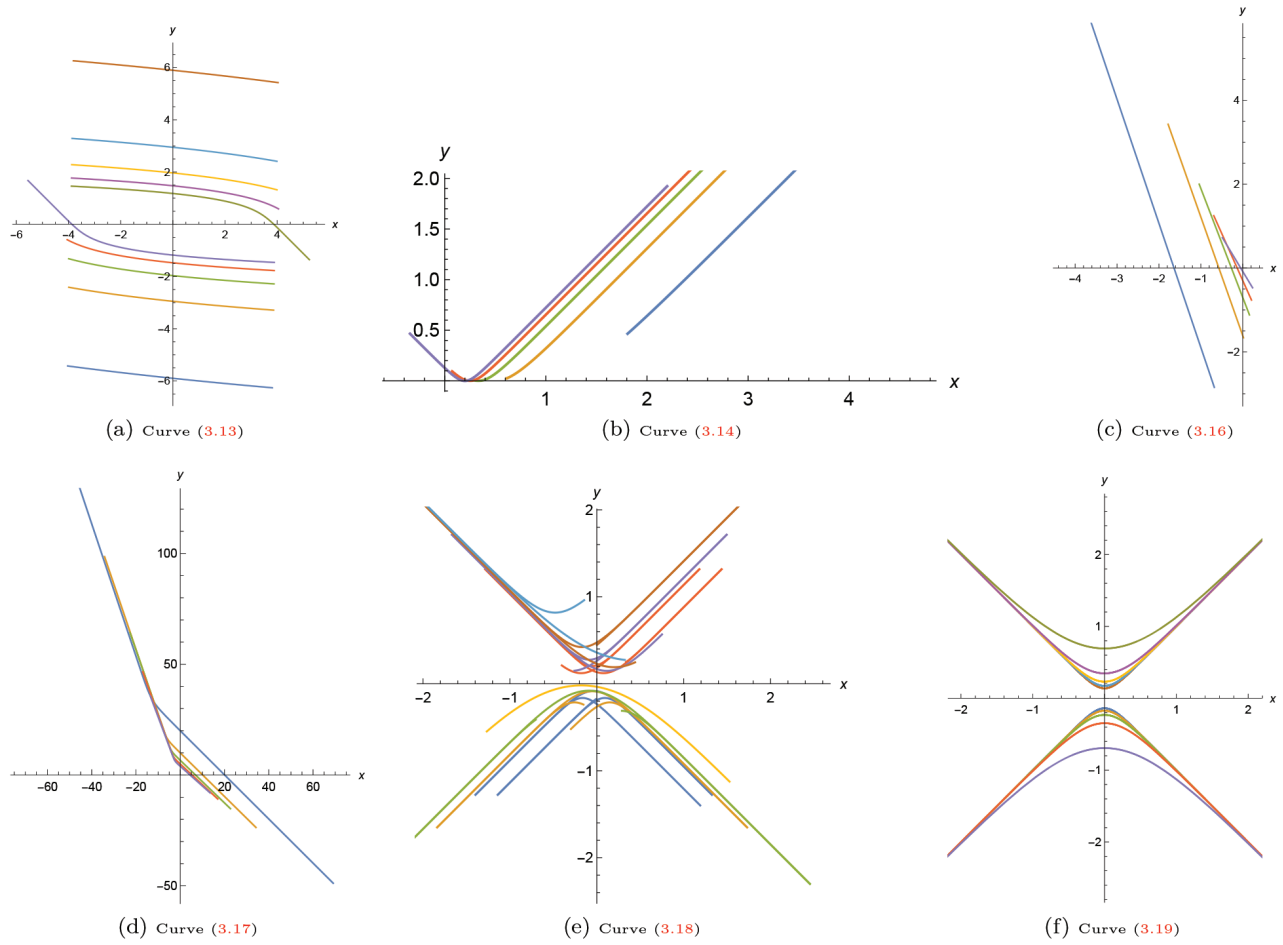
**Theorem 3.4** *The timelike curve  $\beta(u)$  with constant weighted curvature  $\kappa_\varphi = \lambda$  in Lorentz-Minkowski plane with density  $e^{by}$  for  $\lambda = b$  is*

$$\beta(u) = \left( -\frac{bu + \ln \left| \frac{h_6+bu}{2+h_6+bu} \right|}{b} + h_7, -\frac{\ln |(h_6+bu)(2+h_6+bu)|}{b} + h_8 \right), \tag{3.14}$$

*where  $bu + h_6 \in (-2, 0)$  and  $h_i \in \mathbb{R}, i = 6, 7, 8$ .*

Figure 5 (b) shows this curve for  $\lambda = b = 1, 2, 3, 4, 5$  and  $h_6 = h_7 = h_8 = 0$ .

**Theorem 3.5** *The timelike curves with constant weighted curvature  $\kappa_\varphi = \lambda$  in Lorentz-Minkowski plane with density  $e^{by}$  for  $\frac{\lambda}{b} < -1$  are*



**Figure 5.** The timelike curves with constant weighted curvature in Lorentz-Minkowski plane with density  $e^{by}$  for different values of constant  $b$ .

*i) the straight line of*

$$\beta(u) = \left( \left( \frac{-t \pm p + \frac{1}{-t \pm p}}{2} \right) u + h_9, \left( \frac{-t \pm p - \frac{1}{-t \pm p}}{2} \right) u + h_{10} \right), \quad (3.15)$$

whose slope is  $\frac{(-t \pm p)^2 - 1}{(-t \pm p)^2 + 1}$  and  $h_i \in \mathbb{R}$ ,  $i = 9, 10$ ;

*ii)*

$$\beta(u) = \left( \begin{aligned} & \frac{-u}{2} \left( t + p + \frac{1}{t+p} \right) - \left( \frac{\ln\left(\frac{e^{-bpu} - e^{h_{11}}}{(p-t)e^{-bpu} + (p+t)e^{h_{11}}}\right)}{b} \right) + h_{12}, \\ & \frac{-u}{2} \left( t + p - \frac{1}{t+p} \right) - \left( \frac{\ln\left(\frac{e^{-bpu} - e^{h_{11}}}{(p-t)e^{-bpu} + (p+t)e^{h_{11}}}\right)}{b} \right) + h_{13} \end{aligned} \right), \quad (3.16)$$

where  $(1 - e^{h_{11} + bpu})$  and  $((p + t)e^{bpu + h_{11}} + p - t)$  have same signs and  $h_i \in \mathbb{R}$ ,  $i = 11, 12, 13$  or

iii)

$$\beta(u) = \left( \begin{array}{l} \frac{-u}{2} \left( t + p + \frac{1}{t+p} \right) - \left( \frac{\ln(e^{-bp^u} + e^{h_{11}}) - \ln|(p-t)e^{-bp^u} - (p+t)e^{h_{11}}|}{b} \right) + h_{14}, \\ \frac{-u}{2} \left( t + p - \frac{1}{t+p} \right) - \left( \frac{\ln(e^{-bp^u} + e^{h_{11}}) + \ln|(p-t)e^{-bp^u} - (p+t)e^{h_{11}}|}{b} \right) + h_{15} \end{array} \right), \quad (3.17)$$

where  $h_i \in \mathbb{R}$ ,  $i = 11, 14, 15$ . Here, we take  $p = \sqrt{\frac{\lambda^2}{b^2} - 1}$ ,  $t = \frac{\lambda}{b}$ .

Figure 5 (c) shows the curve (3.16) for  $\lambda = 6$ ,  $b = -1, -2, -3, -4, -5$ ,  $h_{11} = -61$  and  $h_{12} = h_{13} = 0$  and Figure 5 (d) shows the curve (3.17) for  $\lambda = 11$ ,  $b = -1, -2, -3, -4, -5$ ,  $h_{11} = 10$  and  $h_{14} = h_{15} = 0$ .

**Theorem 3.6** *The timelike curve with constant weighted curvature  $\kappa_\varphi = \lambda$  in Lorentz-Minkowski plane with density  $e^{by}$  for  $\frac{\lambda}{b} > 1$  is given by (3.16).*

**Theorem 3.7** *The timelike curve  $\beta(u)$  with constant weighted curvature  $\kappa_\varphi = \lambda$  in Lorentz-Minkowski plane with density  $e^{by}$  for  $-1 < \frac{\lambda}{b} < 1$  is*

$$\beta(u) = \left( \begin{array}{l} -\frac{t(h_{16} + 2bu) + 2 \ln|\cos(\frac{q(bu+h_{16})}{2})| - 2 \ln|(t \cos(\frac{q(bu+h_{16})}{2}) - q \sin(\frac{q(bu+h_{16})}{2}))|}{2b} + h_{17}, \\ \frac{th_{16} - 2 \ln|\cos(\frac{q(bu+h_{16})}{2})| - 2 \ln|(t \cos(\frac{q(bu+h_{16})}{2}) - q \sin(\frac{q(bu+h_{16})}{2}))|}{2b} + h_{18} \end{array} \right), \quad (3.18)$$

where  $\tan(\frac{q(bu+h_{16})}{2}) > \frac{t}{q}$ ,  $q = \sqrt{1 - \frac{\lambda^2}{b^2}}$ ,  $t = \frac{\lambda}{b}$  and  $h_i \in \mathbb{R}$ ,  $i = 16, 17, 18$ .

Figure 5 (e) shows this curve for  $\lambda = 1$ ,  $b = -5, -4, -3, -2, 2, 3, 4, 5$ ,  $h_{16} = 1$  and  $h_{17} = h_{18} = 0$ .

So, taking  $\lambda = 0$  in (3.18), we have:

**Corollary 3.8** *The timelike curve  $\beta(u)$  with vanishing weighted curvature  $\kappa_\varphi$  in Lorentz-Minkowski plane with density  $e^{by}$  is given by*

$$\beta(u) = \left( \begin{array}{l} \frac{\ln|\tan(\frac{bu+h_{16}}{2})|}{b} + h_{17}, \\ \frac{-\ln|\frac{1}{2} \sin(bu + h_{16})|}{b} + h_{18} \end{array} \right), \quad (3.19)$$

where  $\tan(\frac{q(bu+c_{16})}{2}) > \frac{t}{q}$  and  $h_i \in \mathbb{R}$ ,  $i = 16, 17, 18$ .

Figure 5 (f) shows this curve for  $b = -5, -4, -3, -2, -1, 1, 2, 3, 4, 5$ ,  $h_{16} = 1$  and  $h_{17} = h_{18} = 0$ .

### 3.3. The case of "a ≠ 0, b = 0"

In this case, the equation (3.3) can be rewritten as

$$f'(u) + a \sinh(f(u)) = \lambda. \quad (3.20)$$

So, from (3.20), we have

$$2f'(u)e^{f(u)} = -a \left[ \left( e^f - \frac{\lambda}{a} \right)^2 - \left( \frac{\lambda^2}{a^2} + 1 \right) \right]. \quad (3.21)$$

Now, we can find the timelike curves by solving the equation (3.21) according to cases of constant  $\lambda$ .

So, we have the following Theorems which can be obtained with the same procedure in Subsection 2.3.

**Theorem 3.9** *The timelike curves with vanishing weighted curvature  $\kappa_\varphi$  in Lorentz-Minkowski plane with density  $e^{ax}$  are*

i) *the straight line of*

$$\beta(u) = (u + g_1, g_2), \tag{3.22}$$

whose slope of lines is 0, i.e. the lines are horizontal and  $g_i \in \mathbb{R}$ ,  $i = 1, 2$ ;

ii)

$$\beta(u) = \left( \frac{\ln(e^{2au} - e^{2g_3})}{a} - u + g_4, \frac{\ln\left(\frac{e^{au} - e^{g_3}}{e^{au} + e^{g_3}}\right)}{a} + g_5 \right), \tag{3.23}$$

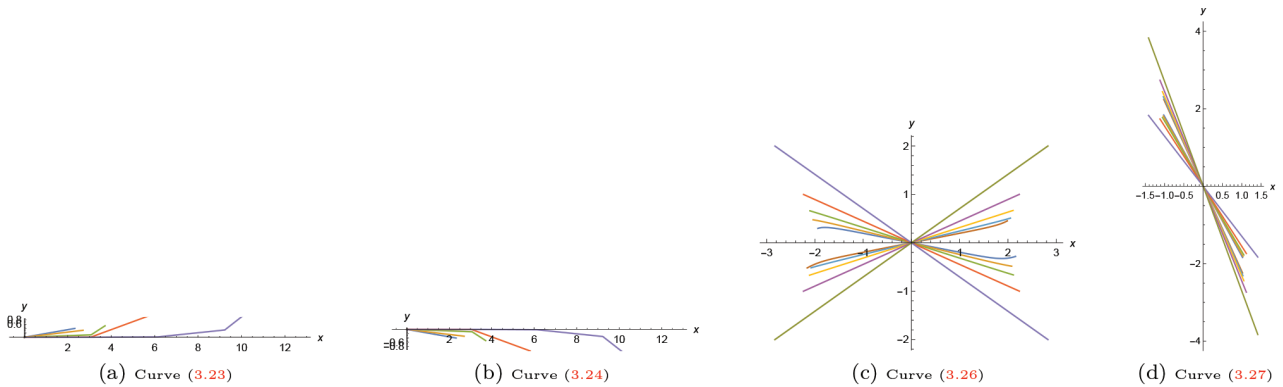
where  $au > g_3$  and  $g_i \in \mathbb{R}$ ,  $i = 3, 4, 5$  or

iii)

$$\beta(u) = \left( \frac{\ln(e^{2au} - e^{2g_3})}{a} - u + g_6, \frac{\ln\left(\frac{e^{au} + e^{g_3}}{e^{au} - e^{g_3}}\right)}{a} + g_7 \right), \tag{3.24}$$

where  $au > g_3$  and  $g_i \in \mathbb{R}$ ,  $i = 3, 6, 7$ .

Figure 6 (a) and (b) show the curves (3.23) and (3.24), respectively, for  $a = 1, 2, 3, 4, 5$ ,  $g_3 = -11$  and  $g_4 = g_5 = g_6 = g_7 = 0$ .



**Figure 6.** The timelike curves with constant weighted curvature in Lorentz-Minkowski plane with density  $e^{ax}$  for different values of constant  $a$

**Theorem 3.10** *The timelike curves with non-zero constant weighted curvature  $\kappa_\varphi = \lambda$  in Lorentz-Minkowski plane with density  $e^{ax}$  are*

i) *The straight line of*

$$\beta(u) = \left( \left( \frac{v + n + \frac{1}{v+n}}{2} \right) u + g_8, \left( \frac{v + n - \frac{1}{v+n}}{2} \right) u + g_9 \right), \tag{3.25}$$

whose slope of lines is  $\frac{(v+n)^2-1}{(v+n)^2+1}$  and  $g_i \in \mathbb{R}$ ,  $i = 8, 9$ ;

ii)

$$\beta(u) = \left( \begin{array}{l} \frac{u}{2} \left( \frac{1}{n-v} + n - v \right) + \frac{1}{a} (\ln(e^{avu} - e^{g_{10}}) + \ln(e^{g_{10}}(n-v)^2 + e^{avu})) + g_{11}, \\ \frac{u}{2} \left( n - v - \frac{1}{n-v} \right) + \frac{1}{a} (\ln(e^{avu} - e^{g_{10}}) - \ln(e^{g_{10}}(n-v)^2 + e^{avu})) + g_{12} \end{array} \right), \quad (3.26)$$

where  $avu > g_{10}$  and  $g_i \in \mathbb{R}$ ,  $i = 10, 11, 12$  or

iii)

$$\beta(u) = \left( \begin{array}{l} \frac{u}{2} \left( \frac{1}{n-v} + n - v \right) + \frac{1}{a} (\ln(e^{avu} + e^{g_{10}}) + \ln(e^{avu} - e^{g_{10}}(n-v)^2)) + g_{13}, \\ \frac{u}{2} \left( n - v - \frac{1}{n-v} \right) + \frac{1}{a} (\ln(e^{avu} + e^{g_{10}}) - \ln(e^{avu} - e^{g_{10}}(n-v)^2)) + g_{14} \end{array} \right), \quad (3.27)$$

where  $e^{avu-g_{10}} > \frac{n-v}{n+v}$  and  $g_i \in \mathbb{R}$ ,  $i = 10, 13, 14$ . Here we take  $v = \sqrt{\frac{\lambda^2}{a^2} + 1}$  and  $n = \frac{\lambda}{a}$ .

Figure 6 (c) and (d) show the curves (3.26) and (3.27), respectively, for  $\lambda = 1$ ,  $a = -5, -4, -3, -2, -1, 1, 2, 3, 4, 5$ ,  $g_{10} = -11$  and  $g_{11} = g_{12} = g_{13} = g_{14} = 0$ .

#### 4. Conclusion and future work

In this study, we have obtained the weighted curvature  $\kappa_\varphi$  of spacelike and timelike curves in Lorentz-Minkowski plane with density  $e^{ax+by}$  and we have given the parametric expressions of the spacelike and timelike curves with constant weighted curvature for the cases of "a = b", "a ≠ 0, b = 0" and "a = 0, b ≠ 0". Also, we have construct some examples of obtained curves for different values of constants a and b.

We hope that, this study will bring a new viewpoint and break fresh ground to geometers who are dealing with the curves in a plane with density. And in the near future, spacelike and timelike curves in Lorentz-Minkowski plane with different densities can be investigated by geometers.

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